^AAA‘ ‘ ‘AAA 过程

吴(工(§§§" " "

éééé ²外²LLLnńń´ ŒÆ

777kkk Æ

111 8ÙÙ ‘ ‘ ÅÅÅ ©©©

8.1 关u ‘ Å i Ä È ©

8.2 关u Brown$ Ä È ©

8.3 ItôÈ © 过程

8.4 Itôú ª

8.5 ‘ Å 微©• 程

8.5 Black-Scholes .

Ù 8 ´ Ú \ 关u Brown$ Ä È © § ? Ø Ù 5 Ÿ ¿ ‰出3 ‘ Å © 析9 7 K Æ¥ k X - ‡ A^ Itôú ª .

x8.1 关u ‘ Å i Ä È © ©

我, I ? Ø 关u { ü ‘ Å i Ä È © m © . $X_1, X_2,$ ´ Õ 立 ‘ Å C 量§ Pf$X_i = 1$g $= P$f$X_i =$ 1g $= \frac{1}{2}$§ - S_nL « 相A i Ä

$$S_n = X_1 + X_2 + \quad + X_n$$

我，Œ± ù w ù | Õ 立‘ Å C 量§ X_n为1 ng ú ² Ù Æ (果($X_n = 1$为I 1 § $X_n =$ 1为Ñ K 1). F$_n = \sigma(X_1, \quad , X_n)$(d f$X_i, 1 \quad i \quad n$g) 成 σ“ ê)§ • Œ± 理) 为包含 $X_1, \quad , X_n$ & 息.- B_n´ F$_n$ $_1$Œÿ ‘ Å C 量S § ' X § L « 1 ng Ù Æž ¤ 下Ù 5 § K § • U 利^ 1 n 1g 9 ± 前 & 息§ Ø U 利^ 1 ng Ù Æ (果.u ´ ž • n Â Ã Z_n为

$$Z_n = \sum_{i=1}^{n} B_i X_i = \sum_{i=1}^{n} B_i (S_i \quad S_i \quad _1) = \sum_{i=1}^{n} B_i \Delta S_i$$

ù 里$\Delta S_i = S_i \quad S_i \quad _1$, $S_0 = 0$. 我，称Z_n为B_n关u S_n ÈÈ ◎◎◎ .

N´ w出fZ_ng´ 关u F_n 鞅§ = é u ¤ k n

$$E[Z_{n+1}|F_n] = Z_n$$

A O/ ，$E[Z_n] = 0$. d 外§ X 果b ½$E[B_n^2] < 1$§ K

$$Var[Z_n] = E[Z_n^2] = \sum_{i=1}^{n} E[B_i^2]$$

¯¢ þ §

$$Z_n^2 = \sum_{i=1}^{n} B_i^2 X_i^2 + 2 \sum_{1 \quad i<j \quad n} B_i B_j X_i X_j$$

2 5 ¿ $X_i^2 = 1$§ X 果$i < j$§ K B_i, X_i, B_jÑ´ F_{j-1}Œÿ § 且X_jÕ 立u F_{j-1}§

$$E[B_i B_j X_i X_j] = E[E[B_i B_j X_i X_j|F_{j-1}]] = E[B_i B_j X_i E(X_j)] = 0$$

x8.2 关u Brown$$$ÄÄ ÈÈÈ©©©

! ½Â 关u Brown$ Ä È © $\int_0^T X(t)dB(t)$ (½{ P 为$\int XdB$)§ ù 里f$B(t)$g´ ~ 维I O Brown$ Ä § k ž • P 为fW_tg. Ä 先• Ä ~ ‡ š ‘ Å { ü 过程f$X(t)$g§ = $X(t)$´ ~ ‡ { ü 函ê (Ø• 赖u $B(t)$). d { ü 函ê ½Â § • 3 $[0,T]$ ©• $0 = t_0 < t_1 <$ $< t_n = T$9 常 ê $c_0, c_1,$ $, c_{n\ 1}$, ¦

$$X(t) = \begin{cases} c_0, & \text{X 果} t = 0 \\ c_i, & \text{X 果} t_i < t \quad t_{i+1},\ i = 0, 1, \quad , n \quad 1 \end{cases}$$

½L « 为

$$X(t) = c_0 I_0(t) + \sum_{i=0}^{n \quad 1} c_i I_{(t_i, t_{i+1}]}(t) \qquad (8.2.1)$$

u ´ § Œ½Â Ù È © 为

$$\int_0^T X(t)dB(t) = \sum_{i=0}^{n \quad 1} c_i [B(t_{i+1}) \quad B(t_i)] \qquad (8.2.2)$$

d Brown$ Ä Õ 立O 量5 Œ• § ú ª (8.2.2)¤ ½Â È ©´ Gauss© Ù ‘ Å C 量§ Ù þ Š 为0§ • 差为

$$Var(\int XdB) = E\left[\sum_{i=0}^{n\ \ 1} c_i(B(t_{i+1}) \quad B(t_i))\right]^2$$

$$= E\left[\sum_{i=0}^{n\ \ 1}\sum_{j=0}^{n\ \ 1} c_i c_j [B(t_{i+1}) \quad B(t_i)][B(t_{j+1}) \quad B(t_j)]\right]$$

$$= \sum_{i=0}^{n\ \ 1}\sum_{j=0}^{n\ \ 1} c_i c_j E[(B(t_{i+1}) \quad B(t_i))(B(t_{j+1}) \quad B(t_j))]$$

$$= \sum_{i=0}^{n\ \ 1} c_i^2(t_{i+1} \quad t_i)$$

^ 4 限 •{ Œ± ò ù ˜ ½Â í 广 ˜ 般 š ‘ Å 函 ê $X(t)$. ´ 我, ‡ ½Â ´ ‘ Å 过程 È © § Ï d ò{ ü 函ê ¥ 常ê c_i^ ‘ Å C 量ξ_i来“ O § ¿ ‡ 求ξ_i´ F$_{t_i}$Œÿ . ù 里F$_t = \sigma$f$B(u), 0 \quad u \quad t$g.u ´ ,d Brown$ Ä 鞅5 Ÿ

$$E[\xi_i(B(t_{i+1}) \quad B(t_i))\text{j}F_{t_i}] = \xi_i E[B(t_{i+1}) \quad B(t_i)\text{j}F_{t_i}] = 0$$

Ï d

$$E[\xi_i(B(t_{i+1}) \quad B(t_i))] = 0.$$

$t > \frac{1}{2}$ž §

$$[Y,Y](t) = \lim \sum_{i=0}^{n\ \ 1} (Y(t_{i+1}^n) \quad Y(t_i^n))^2$$

$$= \xi_0^2 \lim \sum_{t_i<\frac{1}{2}} (B(t_{i+1}^n) \quad B(t_i^n))^2 + \xi_1^2 \lim \sum_{t_i>\frac{1}{2}} (B(t_{i+1}^n) \quad B(t_i^n))^2$$

$$= \xi_0^2 [B,B](\frac{1}{2}) + \xi_1^2 [B,B]((\frac{1}{2},t]) = \int_0^t X^2(s)ds$$

ù 里 4 限Ñ´ δ_n ! 0ž 3 • V Ç Â 敛¿Â 下 4 限. ∎

é 关u Ó˜ ‡ Brown$ Ä f$B(t)$g 两‡ Ø Ó ItðÈ © $Y_1(t) = \int_0^t X_1(s)dB(s)$和 $Y_2(t) = \int_0^t X_2(s)dB(s)$§ d u $Y_1(t) + Y_2(t) = \int_0^t (X_1(s) + X_2(s))dB(s)$§ 我, Œ± ½ Â Y_1和Y_2 ggg ŒŒ 差µ

$$[Y_1, Y_2](t) = \frac{1}{2}([Y_1+Y_2, Y_1+Y_2](t) \quad [Y_1, Y_1](t) \quad [Y_2, Y_2](t))$$

d ½理8.3.6§ k

$$[Y_1, Y_2](t) = \int_0^t X_1(s)X_2(s)ds.$$

x8.4 Itôúúá ª

Itôú ª § ‘ Å © 析¥ C 量O † ú ª ½ 链£ { K § ´ ‘ Å © 析¥ ˜ ‡ Ì ‡ ó ä § N õ - ‡ ú ª § 例X Dynkinú ª § Feynman-Kacú ª ± 9 © Ü È © ú ª § Ñ ´ d Itôú ª 出 .

Ï 为Brown$ Ä 3 $[0,t]$þ g C 差为t§ = 3 • V Ç Â 敛 ¿Â 下

$$\lim_{\delta_n \,!\, 0} \sum_{i=0}^{n \quad 1} (B(t_{i+1}^n) \quad B(t_i^n))^2 = t$$

ù 里ft_i^ng´ $[0,t]$ © • § $\delta_n = \max_{0 \quad i \quad n \quad 1}(t_{i+1}^n \quad t_i^n)$.

þ ª I / ª þ ŒL « 为

$$\int_0^t (dB(s))^2 = \int_0^t ds = t$$

½

$$(dB(t))^2 = dt$$

• ˜ 般/ § 我, k ± 下½理.

‰ 理 8.4.1 设g是k 界ë Y 函数, ft_i^ng是$[0,t]$ © 割, 则é ? 何θ_i^n 2 $(B(t_{i+1}^n), B(t_i^n))$ (= $B(t_i^n), B(t_{i+1}^n)$ƒ m ? ¿ Š), • 概率收ñ ¿Â 下 4 限

$$\lim_{\delta_n \,!\, 0} \sum_{i=0}^{n \quad 1} g(\theta_i^n)(B(t_{i+1}^n) \quad B(t_i^n))^2 = \int_0^t g(B(s))ds \tag{8.4.1}$$

y ² µ Ä 先 $\theta_i^n = B(t_i^n)$§ d g 连Y5 和È © ½ Â § k

$$\sum_{i=0}^{n\ \ 1} g(B(t_i^n))(t_{i+1}^n \quad t_i^n) \ ! \ \int_0^t g(B(s))ds \qquad (8.4.2)$$

我, y ²

$$\sum_{i=0}^{n\ \ 1} g(B(t_i^n))(B(t_{i+1}^n) \quad B(t_i^n))^2 \quad \sum_{i=0}^{n\ \ 1} g(B(t_i^n))(t_{i+1}^n \quad t_i^n) \ ! \ 0 \qquad (8.4.3)$$

3 L^2¥ 成立. P $\Delta B_i = B(t_{i+1}^n) \quad B(t_i^n)$, $\Delta t_i = t_{i+1}^n \quad t_i^n$§ K d Brown$ Ä Õ 立O 量5 和 ^ ‡ Ï 望 •{

$$E\left[\sum_{i=0}^{n\ 1} g(B(t_i^n))((\Delta B_i)^2 \quad \Delta t_i)\right]^2$$

$$= E\left[E\left(\sum_{i=0}^{n\ 1} g^2(B(t_i^n))((\Delta B_i)^2 \quad \Delta t_i)^2 \mathrm{j} \mathsf{F}_{t_i}\right)\right]$$

$$= E\left[\sum_{i=0}^{n\ 1} g^2(B(t_i^n))E\left(((\Delta B_i)^2 \quad \Delta t_i)^2 \mathrm{j} \mathsf{F}_{t_i}\right)\right]$$

$$= 2E\left[\sum_{i=0}^{n\ 1} g^2(B(t_i^n))(\Delta t_i)^2\right]$$

$$2\delta_n E[\sum_{i=0}^{n\ 1} g^2(B(t_i^n))\Delta t_i] \ ! \ 0, \quad (\quad \delta_n \ ! \ 0\check{z} \ )$$

Ï d § 3 þ • Â 敛 ¿Â 下

$$\sum_{i=0}^{n\ \ 1} g(B(t_i^n))((\Delta B_i)^2 \quad \Delta t_i) \ ! \ 0$$

ù § $\sum_{i=0}^{n\ \ 1} g(B(t_i^n))(B(t_{i+1}^n) \quad B(t_i^n))^2$†
$\sum_{i=0}^{n\ \ 1} g(B(t_i^n))(t_{i+1}^n \quad t_i^n)$k 相Ó 4 限$\int_0^t g(B(s))ds.$

é ? ¿ θ_i^n 2 $(B(t_{i+1}^n), B(t_i^n))$§ δ_n ! 0ž §

$$\sum_{i=0}^{n\ \ 1}(g(\theta_i^n) \quad g(B(t_i^n)))(\Delta B_i)^2$$

$$\max_i((g(\theta_i^n) \quad g(B(t_i^n))) \sum_{i=0}^{n\ \ 1}(B(t_{i+1}^n) \quad B(t_i^n))^2$$

(8.4.4)

d g和B 连Y5 § 我, k $\max_i((g(\theta_i^n) \quad g(B(t_i^n))) \; ! \; 0, \; a.s..$
d Brown$ Ä g C 差½ Â $\sum_{i=0}^{n} {}^{1}(B(t_{i+1}^n) \quad B(t_i^n))^2 \; !$
t.

u ´ $\delta_n \; ! \; 0$ž § $\sum_{i=0}^{n} {}^{1}(g(\theta_i^n) \quad g(B(t_i^n)))(\Delta B_i)^2 \; ! \; 0.$
Ï d $\sum_{i=0}^{n} {}^{1} g(\theta_i^n)(\Delta B_i)^2$† $\sum_{i=0}^{n} {}^{1} g(B(t_i^n))(\Delta B_i)^2$ä k 相Ó

- V Ç Â 敛¿Â 4 限 $\int_0^t g(B(s))ds.$ ∎

现3 § 我, ‰出Itô ú ª .

‱ 理 8.4.2 X 果f是 次ë Y Œ微函数, 则é ? 何t, k

$$f(B(t)) = f(0) + \int_0^t f^{0}(B(s))dB(s) + \frac{1}{2}\int_0^t f^{00}(B(s))ds \tag{8.4.5}$$

y ² µ ´ „ ª (8.4.5)¥ È © Ñ ´ · ½ . $[0,t]$ ©

- ft_i^ng§ k

$$f(B(t)) = f(0) + \sum_{i=0}^{n\ \ 1}[f(B(t_{i+1}^n)) \quad f(B(t_i^n))]$$

é $f(B(t_{i+1}^n)) \quad f(B(t_i^n))$A^ Taylorú ª

$$f(B(t_{i+1}^n)) \quad f(B(t_i^n)) = f^0(B(t_i^n))(B(t_{i+1}^n) \quad B(t_i^n))$$
$$+\frac{1}{2}f^{00}(\theta_i^n)(B(t_{i+1}^n) \quad B(t_i^n))^2$$

Ù¥ θ_i^n 2 $(B(t_{i+1}^n), B(t_i^n))$.u ´

$$f(B(t)) = f(0) + \sum_{i=0}^{n \quad 1} f^0(B(t_i^n))(B(t_{i+1}^n) \quad B(t_i^n)) +$$
$$\frac{1}{2}\sum_{i=0}^{n \quad 1} f^{00}(\theta_i^n)(B(t_{i+1}^n) \quad B(t_i^n))^2 \qquad (8.4.6)$$

- δ_n ! 0 4 限§ K ª (8.4.6)¥ 1 ~ ‡ 和Â 敛u ItôÈ © $\int_0^t f^0(B(s))dB(s)$.利^ ½ 理8.4.1Œ• ª (8.4.6)¥ 1 ‡ 和Â 敛u $\int_0^t f^{00}(B(s))ds$. ∎

ª (8.4.5)称为Brown$ Ä Itôú ª § d d w 出§ Brown$ Ä 函ê Œ± L « 为˜ ‡ ItôÈ © \ þ ˜ ‡ ä kk . C 差 ý é 连Y 过程.我，称ù 类过程为Itô过程§ î ， / § 我，k 下¡ ½Â .

8.4.3 如果过程f$Y(t), 0 \quad t \quad T$g可以表示为

$$Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s), \quad 0 \quad t \quad T \tag{8.4.7}$$

其中过程f$\mu(t)$g和f$\sigma(t)$g满足

(1) $\mu(t)$是适应的并且$\int_0^T$ j$\mu(t)$j$dt <$ 1， $a.s.$

(2) $\sigma(t)$ 2 V .

则称f$Y(t)$g为Itô 过程.

k ž 我, • ò Itô过程(8.4.7)P 为微© / ª

$$dY(t) = \mu(t)dt + \sigma(t)dB(t), \quad 0 \quad t \quad T \qquad (8.4.8)$$

Ù ¥ 函ê $\mu(t)$称为¤ £ 系ê ,$\sigma(t)$称为扩Ñ 系ê ,§ , Œ± • 赖u $Y(t)$½ $B(t)$§ $ – ‡ ´ » f$B(s), s \quad t$g§ (例 X $\mu(t) = \cos(M(t) + t)$§ ù 里 $M(t) = \max_{s \ t} B(s)$. ˜ 类 š 常- ‡ 情/ ´ μ† $\sigma(t)$ == Ï 过$Y(t)$• 赖u t§ 3ù « 情¹ 下§ ª (8.4.8)A U 为

$$dY(t) = \mu(Y(t))dt + \sigma(Y(t))dB(t), \quad 0 \quad t \quad T \qquad (8.4.9)$$

X 果^ 微© / ª L « § Itôú ª (8.4.5)C 为

$$d(f(B(t)) = f^{0}(B(t))dB(t) + \frac{1}{2}f^{00}(B(t))dt \qquad (8.4.10)$$

例 8.4.4 求$d(e^{B(t)})$.

))) é 函ê $f(x)=e^x$^ Itðú ª § d ž $f^0(x)=e^x, f^{00}(x)=e^x$§ ¤ ±

$$d(e^{B(t)})=e^{B(t)}dB(t)+\frac{1}{2}e^{B(t)}dt$$

u ´ ，$X(t)=e^{B(t)}$ä k ‘ Å 微© / ª

$$dX(t)=X(t)dB(t)+\frac{1}{2}X(t)dt$$

3 7 K A^ ¥ § 股¦ d ， $S(t)$´ ^ ‘ Å 微©• 程$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$£ ã .X 果$a(t)$L « 3 ž • tÝ] ö 股¦ ˆ 股Â Ã § @o 3 ‡ ž m区m$[0, T]$S Â Ã 为

$$\int_0^T a(t)dS(t) = \mu \int_0^T a(t)S(t)dt + \sigma \int_0^T a(t)S(t)dB(t)$$

下¡ ½理‰出 关u Itô过程 Itôú ª

‰ 理 8.4.5　设 f $X(t)$ g是 d

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

给出　Itô过程，$g(t,x)$是$[0,\ 1)$　$\mathbb{R}$þ　　次ë Y Œ微函数.则

$$\mathrm{f}Y(t) = g(t, X(t))\mathrm{g}$$

E 为Itô过程，并且

$$dY(t) = \frac{\partial g}{\partial t}(t,X(t))dt + \frac{\partial g}{\partial x}(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X(t))\ \ (dX(t))^2 \qquad (8.4.11)$$

Ù ¥ $(dX(t))^2 = (dX(t))\ \ (dX(t))$按照下面规则O 算:

$dt\ \ dt = dt\ \ dB(t) = dB(t)\ \ dt = 0,\ \ dB(t)\ \ dB(t) = dt$

= (8.4.11)Œ± 改　为

$$dY(t) = \left(\frac{\partial g}{\partial t}(t, X(t)) + \frac{\partial g}{\partial x}(t, X(t))\mu(t)\right) dt + \frac{\partial g}{\partial x}(t, X(t))\sigma(t)dB(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))\sigma^2(t)dt$$

特别/ ，X 果$g(t,x) = g(x)$• 是x　函数，(8.4.12){z 为

$$dY(t) = [g^0(X(t))\mu(t) + \frac{1}{2}g^{00}(X(t))\sigma^2(t)]dt + g^0(X(t))\sigma(t)dB(t) \quad (8.4.12)$$

例 8.4.6 á Ï | 率模. （ Vasicek model ）d X 下

- 程给出

$$dR_t = a(b \quad R_t)dt + \sigma dB_t, \quad R_0 = r.$$

| ^ IT O公式O 算$d(R_t e^{at})$，并解出R_t.