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问题一：

ф$W(t), t \geq 0$ Ό ظ $\mathcal{F}_t, t \geq 0$ ỉ Ό ظ σ ╙ - $W^2(t) - t$ ĩ

参考答案： $0 \leq s \leq t$,

$$E[W^2(t) - t|\mathcal{F}(s)]$$

$$= E[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t|\mathcal{F}(s)]$$

$$= \mathrm{E}[(W(t) - W(s))^2 - t|\mathcal{F}(s)] + E[2W(t)W(s)|\mathcal{F}(s)] + E[-W^2(s)|\mathcal{F}(s)]$$

$$= [(t - s) - t] + 2W(s)E[W(t)|\mathcal{F}(s)] - W^2(s)$$

$$= W^2(s) - s$$

问题二：

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Ɣ ₦ ' $p < 1/2$ ₦ ' $1 - p$ $P(\xi_n = 1) = p$, $P(\xi_n = -1) = 1 - p$ ʟ n ℓ ɹ S_n $S_n = S_0 + \sum_{i=1}^{n} \xi_i$ $S_0 > 0$ ˜

ɹ T ˜ ҿ ▫ 0 Ɏ Ҡ ڗ 0 ặ N $T = \inf\{n: S_n = 0$ ặ $N\}$ ˜ σ- ╙ $F_n = \sigma(S_1, S_2, \dots, S_n)$

(1) - $M_n = \left(\frac{1-p}{p}\right)^{S_n}$

(2) - T

(3) ĩ ת ∂ $P(S_T = 0)$ $P(S_T = N)$

(4) - $Y_n := S_n + n(1 - 2p)$

(5) պ Ɣ $E(T)$

参考答案：：

(1) $\mathrm{E}[\mathrm{M}_{n+1}|\mathcal{F}(\mathrm{n})]$

$$= E\left[\left(\frac{1-p}{p}\right)^{S_{n+1}}\middle|\mathcal{F}(\mathrm{n})\right]$$

$$= \left(\frac{1-p}{p}\right)^{S_n} E\left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}}\middle|\mathcal{F}(\mathrm{n})\right]$$

$$= \left(\frac{1-p}{p}\right)^{S_n}\left[\frac{1-p}{p}\times p+\left(\frac{1-p}{p}\right)^{-1}\times(1-p)\right]$$

$$= \left(\frac{1-p}{p}\right)^{S_n}$$

$$= M_{\mathrm{n}}$$

So martingale.

(2) $\mathrm{T}=\inf\{\mathrm{n}: \mathrm{S}_n=0 \text{ or } \mathrm{S}_n=\mathrm{N}\}$,

so $\{\mathrm{T}=\mathrm{k}\}=\{\mathrm{S}_{\mathrm{i}}\neq 0\&\mathrm{N}, \text{for all } \mathrm{i}=0,1,\ldots \mathrm{k}-1\}\cap\{\mathrm{S}_k=0 \text{ or } \mathrm{S}_k=\mathrm{N}\}\in\mathcal{F}(\mathrm{n})$, so T is a stopping time.

(3) Since M_{n} is a martingale, T is a stopping time,

a. $\mathrm{P}\{\mathrm{T}<\infty\}=1-\mathrm{P}\{\mathrm{T}=+\infty\}=1-\lim\limits_{n\to\infty}\frac{1}{2^n}=1$

b. $\mathrm{E}[|\mathrm{M}_{\mathrm{n}}|]\leq(\frac{1-p}{p})^N<+\infty$

c. $\lim\limits_{n\to\infty}\mathrm{E}[|\mathrm{M}_{\mathrm{n}}|\cdot]I_{T\geq n}]\leq\lim\limits_{n\to\infty}(\frac{1-p}{p})^N\times P\{T>n\}=0$

Therefore, $\mathrm{E}[\mathrm{M}_T]=\mathrm{E}[\mathrm{M}_0]$

$$\Rightarrow\begin{cases}\left(\frac{1-p}{p}\right)^{\mathrm{N}}P\{\mathrm{S}_T=N\}+\left(\frac{1-p}{p}\right)^{0}P\{\mathrm{S}_T=0\}=\left(\frac{1-p}{p}\right)^{\mathrm{S}_0}\\ P\{\mathrm{S}_T=N\}+P\{\mathrm{S}_T=0\}=1\end{cases}$$

$$\Rightarrow \mathrm{P}\{\mathrm{S}_T=N\}=\frac{\left(\frac{1-p}{p}\right)^{\mathrm{S}_0}-1}{\left(\frac{1-p}{p}\right)^{\mathrm{N}}-1}\quad and\quad P\{\mathrm{S}_T=0\}=1-P(S_T=N)$$

(4) $\mathrm{E}[\mathrm{Y}_{\mathrm{n}}|\mathcal{F}(\mathrm{n}-1)]$

$$=\mathrm{E}[\{\mathrm{S}_n+\mathrm{n}(1-2\mathrm{p})|\mathcal{F}(\mathrm{n}-1)]$$

$$=\mathrm{E}[\mathrm{S}_n|\mathcal{F}(\mathrm{n}-1)]+n(1-2p)$$

$= S_{n-1} + E[\xi_n] + n(1-2p)$

$= S_{n-1} + p - 1 + n(1-2p)$

$= S_{n-1} + (n-1)(1-2p)$

$= Y_{n-1}$

(5) S i n Y_n e i s a m a r t i n g a l e , T $E[Y_T] = E[Y_0]$, i n i t e s t o p p i

$\Longrightarrow E[S_T + T(1-2p)] = E[S_0] = S_0,$

$$\Longrightarrow E[T] = \frac{S_0 - N \times P\{S_T = N\} - 0 \times P\{S_T = 0\}}{1-2p}$$

$$= \left. S_0 - N \times \frac{\left(\frac{1-p}{p}\right)^{S_0} - 1}{\left(\frac{1-p}{p}\right)^{N} - 1} \middle/ 1-2p \right.$$

问题三：

$\xi_{n,k}, n = 1,2,,\ldots,k = 1,2,\ldots.$

$\mu = E(\xi_{n,k}), \sigma^2 = Var(\xi_{n,k})$ Galton-Watson branching

$$Z_0 := 1, Z_{n+1} := \sum_{k=1}^{Z_n} \xi_{n+1,k}$$

σ $\mathcal{G}_n := \sigma(Z_j : 0 \le j \le n), n = 0,1,2,\ldots$

$M_n := \mu^{-n} Z_n, n = 1,2,\ldots.$

参考答案：

$E[M(n+1)|\mathcal{F}(n)]$

$= \mu^{-(n+1)} E\left(\sum_{k=1}^{Z_n} \xi_{n+1,k} \middle| \mathcal{G}(n)\right)$

$$= \mu^{-(n+1)} \sum_{k=1}^{Z_n} E(\xi_{n+1,k} | \mathcal{G}(n))$$

$= \mu^{-(n+1)} Z_n \mu$

$= M_n$

问题四：

ᵴ $W(t), t \geq 0$ ʹO ظ , $W(t)$ $\sigma^2 = 9t$ ' $W(0) = 10$ ∂ P($W(0.5) > 10$)。

参考答案：

At $W(0.5)$, we have $W(0.5) = 10 + 3\sqrt{0.5}Z$, where Z is a standard normal random variable ($m = 0, s^2 = 1$). And so to calculate P($W(0.5) > 10$), we are calculating P($10 + 3\sqrt{0.5}Z > 10$). Simplifying we have

$$P\left(10 + 3\sqrt{0.5}Z > 10\right) = P\left(3\sqrt{0.5}Z > 0\right) = 0.5$$

问题五：

ᵴ $B_1(t)$ $B_2(t)$ ‾ ⱴ ‰ ʹO ظ - $X_t := B_1(t) + B_2(t)$ ʹO ظ

参考答案：：

1 Continuous 2 Normal For s,t $X(t) - X(s) = [B_1(t) - B_1(s)] - [B_2(t) - B_2(s)]$ is stationary increment